PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON THURSDAY, FEBRUARY 1, 2024, ON GEOMETRIC SERIES, ON THE TEST FOR DIVERGENCE, ON p-series, and on the INTEGRAL TEST FOR CONVERGENCE OR DIVERGENCE

CLASS # 6

RECALL FROM TUESDAY

For $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (0.1)^n$ = the S.O.P.S. of the Sequence of Terms $\{(0.1)^n\}_{n=1}^{\infty}$ = "0.1, 0.01, 0.001, ..."

$$\sum_{n=1}^{\infty}(0.1)^n = \{S_n\}_{n=1}^{\infty} \begin{cases} S_1 = 0.1 \\ S_2 = 0.11 \\ S_3 = 0.111 \\ S_4 = 0.1111 \\ \vdots \\ \text{etc} \end{cases}$$

The Series $\sum_{n=1}^{\infty} (0.1)^n$ = $\overbrace{\text{"}0.1, 0.11, 0.111, 0.1111, \ldots\text{"}}^{\text{The Sequence of Partial Sums}}$

and $\lim_{n\to\infty} S_n = \frac{1}{9}$.

When $\lim_{n\to\infty} S_n$ exists, we say "The Series $\sum_{n=1}^{\infty} a_n$ is Convergent."

when $\lim_{n\to\infty} S_n$ D.N.E., "The Series $\sum_{n=1}^{\infty} a_n$ is Divergent"

NOTATION: We sometimes write $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$ to mean $\sum_{n=1}^{\infty} a_n = S_1, S_2, S_3, S_4, \ldots$

Series $\sum_{n=0}^{\infty} a_n$

The Test for Divergence

Given a series $\sum_{n=1}^{\infty} a_n$, if you consider the sequence of terms $\{a_n\}_{n=1}^{\infty}$ and you find that the $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ D.N.E., then the series $\sum_{n=1}^{\infty} a_n$ is <u>Divergent</u>, i.e. $\lim_{n\to\infty} S_n$ DNE.

Ex: Is the series $\sum_{n=1}^{\infty} \frac{n}{4n+1}$ C or D?

<u>Sol'n</u>: Consider $\left\{\frac{n}{4n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{n}{4n+1} = \lim_{n\to\infty} \frac{\cancel{n}(1)}{\cancel{n}(4+1/n)} = \lim_{n\to\infty} \frac{1}{4+\underbrace{\tfrac{1}{n}}_{\to 0}} = \frac{1}{4} \neq 0$$

Sentence of Justification: Since $\lim_{n\to\infty} \frac{n}{4n+1} = \frac{1}{4}$ and $\frac{1}{4} \neq 0$, the series $\sum_{n=1}^{\infty} \frac{n}{4n+1}$ is Divergent by the Test for Divergence.

Whenever you conclude that a series is C or D, you must write a sentence justifying your conclusion, stating which test you used to make your conclusion.

Def'n: Let a and r be real numbers.

A **Geometric Sequence** is a sequence of the form: $\{ar^n\}_{n=0}^{\infty} =$ "$a, ar, ar^2, ar^3, \ldots$"

$= \{ar^{n-1}\}_{n=1}^{\infty}$

The Common Ratio $= \dfrac{ar^{n+1}}{ar^n} = \underline{\underline{r}}$

Def'n: The Sequence of Partial Sums made from a sequence of terms which a Geometric Sequence, is a **Geometric Series**:

$$\sum_{n=1}^{\infty} ar^{n-1} = \text{"}a + ar + ar^2 + ar^3 + \ldots\text{"}$$

The Rule for Convergence or Divergence of a Geometric Series:

For $\sum_{n=1}^{\infty} ar^{n-1}$, the series is $\begin{cases} \text{Convergent if } -1 < r < 1 \ (|r| < 1) \\ \text{Divergent if } |r| \geq 1 \end{cases}$

(Same for $\sum_{n=0} ar^n$)

and, when $|r| < 1$,

the sum S is given by $S = \sum_{n=1}^{\infty} ar^{n-1} = \dfrac{\text{First Term}}{(1 - \text{Common Ratio})}$.

Consider $\sum_{n=1}^{\infty} (0.1)^n = \sum_{n=1}^{\infty} (0.1)\cdot(0.1)^{n-1} = \sum_{n=1}^{\infty} ar^{n-1}$

It is a Geometric Series with Common Ratio (0.1).

Since $-1 < 0.1 < 1$, $\sum_{n=1}^{\infty} (0.1)^n$ is Convergent by the Test for Geometric Series.

The summation S is

$$S = \frac{\text{FIRST TERM}}{1 - \text{Common Ratio}} = \frac{0.1}{1.0 - 0.1}$$

$$S = \frac{0.1}{0.9} = \frac{1}{9}$$

The Series $\sum_{n=1}^{\infty} (0.1)^n$ is Convergent because it is a Geometric Series with common ratio $r = 0.1$ and $-1 < 0.1 < 1$.

What about $\sum_{n=1}^{\infty} \frac{2^n}{5(3^{n-1})} = \sum_{n=1}^{\infty} \frac{2(2^{n-1})}{5\cdot(3^{n-1})}$

$$= \sum_{n=1}^{\infty} \frac{2}{5}\left(\frac{2}{3}\right)^{n-1} ?$$

It is a geometric series.

Observation:

Given the series, $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots = \{S_n\}_{n=1}^{\infty}$,

Consider these series:

$$\sum_{n=7}^{\infty} a_n = a_7 + a_8 + a_9 + a_{10} + \cdots$$

$$b_1 + b_2 + b_3 + b_4 + \underbrace{a_1 + a_2 + a_3 + \cdots}_{\sum_{n=1}^{\infty} a_n}$$

All of these series are C or D Together.

Coming up, when a condition says, "For all $n \geq 1, \ldots$",
This really means, "For all $n \geq K$ for some $K \geq 1$"

Tests for Convergence of $\sum_{n=1}^{\infty} a_n$

For Now, All Series $\sum_{n=1}^{\infty} a_n$ Considered will have a NON-NEGATIVE SEQUENCE OF TERMS,

i.e., $a_n \geq 0$ for all $n \geq 1$.

THE INTEGRAL TEST For C or D

of $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$ for all $n \geq 1$.

Suppose a_n has the formula $a_n = f(n)$.
[Like; $a_n = \frac{1}{n^5}$], define the function $y = f(x)$ which you get by replacing n with x in the $f(n)$ formula.

[For example, for $\sum_{n=1}^{\infty} \frac{1}{n^5}$, $a_n = \frac{1}{n^5}$, define

$y = f(x) = \frac{1}{x^5}$, "let $f(x) = \frac{1}{x^5}$".]

If the function $f(x)$ satisfies the "FOUR SPECIAL CONDITIONS" (Shown Later),

Then (A) "The Integral Test Applies"

and (B) If the Improper Integral $\int_1^{\infty} f(x)\,dx$

$\int_1^{\infty} f(x)\,dx$ is Convergent (or Divergent),

then the Series $\sum_{n=1}^{\infty} a_n$ is Convergent (or Divergent).

Problem: Consider $\sum_{n=1}^{\infty} \frac{1}{n^5}$. Is it C or D?

Sol'n: Define the function f as $f(x) = \frac{1}{x^5}$

We need to show the Special Conditions

$\underbrace{①, ②, ③}$, ④

Only State that these are true about f.

You must show this me.

① f is continuous on $[1, \infty)$.

② f is positive on $[1, \infty)$

③ $f(n) = \frac{1}{n^5} = a_n$ for all $n \geq 1$.

④ (Two methods to show ④)

method 1

With Algebra:

If $1 \leq a < b$,
then $a^5 < b^5$, so $\frac{1}{a^5} > \frac{1}{b^5}$, so $f(a) > f(b)$,
so, $f(x)$ is decreasing on $[1, \infty)$ (or $[k, \infty)$)

method 2

Using the Derivative $f'(x)$:
By showing $f'(x) < 0$ for all $x \geq 1$. (or k)
Here, $f(x) = x^{-5}$, so, $f'(x) = \frac{-5}{x^6} = \frac{\text{Neg}}{\text{pos}} < 0$ when $x \geq 1$
So, f is Decreasing on $[1, \infty)$

The Integral Test Applies:

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^5}\,dx = \lim_{t\to\infty} \int_1^t x^{-5}\,dx$$

$$= \lim_{t\to\infty} \left(\frac{x^{-4}}{-4}\right)\Bigg]_1^t = \lim_{t\to\infty} \left(-\frac{1}{4x^4}\right)\Bigg|_1^t$$

$$= \lim_{t\to\infty} \left(\left(-\frac{1}{4t^4}\right) - \left(-\frac{1}{4(1)^4}\right)\right)$$

$$= \lim_{t\to\infty} \left(\underbrace{-\frac{1}{4t^4}}_{\to 0} + \frac{1}{4}\right) = \frac{1}{4}$$

So, the Improper Integral $\int_1^\infty \frac{1}{x^5}\,dx$ is C..

sentence of Justification

[Since the Improper Integral $\int_1^\infty \frac{1}{x^5}\,dx$ is Convergent, The Series $\sum_{n=1}^\infty \frac{1}{n^5}$ is Convergent by the Integral Test.]

Def'n: For a fixed positive exponent p, the series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a "p-series".

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

The p-Series Test for C or D

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is $\begin{cases} C \text{ if } p > 1 \\ D \text{ if } 0 < p \leq 1 \end{cases}$

Problem Is $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n})^5}$ C or D?

Sol'n: $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n})^5} = \sum_{n=1}^{\infty} \frac{1}{(n^{\frac{1}{2}})^5} = \sum_{n=1}^{\infty} \frac{1}{n^{(\frac{5}{2})}}$,

which is a p-series with $n = \frac{5}{2} > 1$.

Sentence of JUSTIFICATION: The Series $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n})^5}$ is Convergent because it is a p-series with $p = 5/2$ and $5/2 > 1$, by the p-Series test.